Hamiltonian Formulation of Piano String Lagrangian Density with Riemann-Liouville Fractional Definition

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ABSTRACT

The fractional form of the piano string Lagrangian density is presented using the Riemann-Liouville fractional derivative. Agrawal procedure is employed to obtain equation of motion in fractional form. The Hamiltonian equations of motion resulting from the Piano string Lagrangian density are obtained. Conserved quantities are also derived using Noether's theorem.

KEYWORDS: Piano String Lagrangian Density, Fractional Euler Lagrange Equation, Riemann-Liouville Fractional Definition

1-INTRODUCTION

Most musical instruments produce tones whose partial tones, or over tones, are harmonic: their frequencies are completely multiples of a fundamental frequency. The piano is an exception. The vibrational properties for a piano string can be described by a set of differential and partial differential equations derived from the general law of physics.S uch a set of equations, which defines the instrument with a higher or lesser degree of perfection, often referred to a physical model1-5.

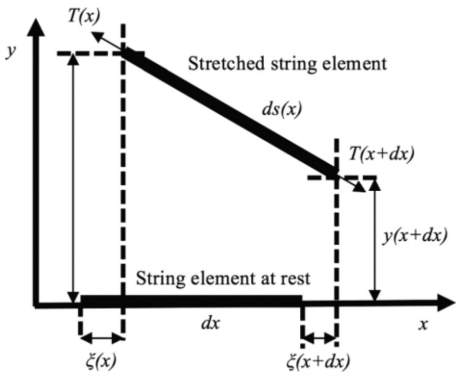
Antoine Chaigne3 started from the fundamental equations of a damped, stiff string interacting with a nonlinear hammer, from which the time histories of a string displaced and velocity from each point of the string are computed in the time domain. Historically, Hiller and Ruiz4were the first to solve the equations of the vibrating string numerically in order to simulate musical sounds.

Harvey Fletcher5developed the equations that govern the vibration of a solid string along traditional lines.When a piano string is displaced a distance y at the position x, the restoring force due to the tension $T$is known to be $T\left(\frac{\partial^2 y}{\partial x^2}\right)$, and the restoring force due to the elastic stiffness is $-QSK^2\left(\frac{\partial^4 y}{\partial x^4}\right)$, where Q is Young's modulus of elasticity, S is the area of cross section of the wire, and K is its radius of gyration. Let σ be the linear density and t the time. Then, the equation governing the motion of the piano string is

$$-QSK^2\left(\frac{\partial^4 y}{\partial x^4}\right)+T\left(\frac{\partial^2 y}{\partial x^2}\right)=\sigma\left(\frac{\partial^2 y}{\partial x^2}\right).$$

This is the form of the equation originally set up by Lord Rayleigh6.

An element of a piano string at equilibrium with length dx and a corresponding element of a stretched piano string with length ds are shown in Fig. 1, where y and n are the transverse and longitudinal displacements of the string, respectively. It was shown that by expanding both y and n as a series of polynomials and truncating at third order, the added force per unit length on the element in the longitudinal direction caused by the transverse displacement is given byF_x



In this work, we will find equations of motion that are subject to the pianostring through the Lagrangian density of the piano string. Then, we will rephrase these equations using the definition of Left and Right Riemann-Liouville fractional derivative. We can also find some other quantities that reflect the saved functions such as Hamiltonian and momentum and other quantities using the definition of Riemann-Liouville fractional derivatives.

2-MATHEMATICAL TOOLS

Several definitions of fractional derivative have been proposed. These definitions include Caputo, Riemann-Liouville, and GrÜnwald-Letenikove and others. Here, we formulate the problem in terms of left and right Riemann-Liouville7-14.

Consider a function $f(x)$, where this function depends on n variable $(x = x_1, x_2, \dots \dots \dots x_n)$, and is defined over the domain $\Omega = [a_1, b_1] \times \dots \dots \times [a_n, b_n]$. The left and right partial Riemann-Liouville fractional derivative of order α_k with respect to x_k, where $0 < \alpha_k < 1$ are defined respectively as[7]:

$$\left({}_{+}\partial_k^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha_k)} \partial x_k \int_{a_k}^{x_k} \frac{f(x_1,\dots,x_{k-1},u,x_{k+1},\dots,x_n)}{(x_k-u)^{\alpha_k}} du \quad (1)$$

$$\left({}_{-}\partial_k^{\alpha} f\right)(x) = \frac{-1}{\Gamma(1-\alpha_k)} \partial x_k \int_{x_k}^{b_k} \frac{f(x_1,\dots,x_{k-1},u,x_{k+1},\dots,x_n)}{(u-x_k)^{\alpha_k}} du \quad (2)$$

The above definitions contain various subscripts and superscripts that need to be made clear. $\partial x_k f$ is the partial derivative of f with respect to the variable x_k, the subscript $(+)$ and $(-)$ indicate the left and right fractional derivative respectively, and the subscript k and the superscript α indicate that the derivative is taken with respect to the variable x_k and it is of order α_k.

The action of the classical field containing fractional partial derivatives takes the form

$$S = \int \mathcal{L}\left(\phi, {}_{-}\partial_k^{\alpha}\phi, {}_{+}\partial_k^{\alpha}\phi\right) d^3x dt \quad (3)$$

extremization of this action leads to the fractional Euler Lagrange equation of the form[10-14]

$$\frac{\partial \mathcal{L}}{\partial \phi} + \left[{}_{+}\partial_k^{\alpha} \left(\frac{\partial \mathcal{L}}{\partial({}_{-}\partial_k^{\alpha}\phi)} \right) + {}_{-}\partial_k^{\alpha} \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_k^{\alpha}\phi)} \right) \right] = 0 \quad (4)$$

for α=1, the last equation reduces to the standard Euler Lagrange

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \quad (5)$$

3-Piano String Lagrangian Density

An elastic Piano string can vibrate both longitudinally and transversely; and the two vibrations influence one another. A Lagrangian that takes into account the lowest order effect of stretching on the local string tension, and can therefore mode this coupled motion, is $\mathcal{L}$

$$\mathcal{L} = \frac{\rho_\circ}{2}\left(\dot{\xi}^2 + \dot{\eta}^2\right) - \frac{\lambda}{2}\left(\frac{\tau_\circ}{\lambda} + \frac{d\xi}{dx} + \frac{\left(\frac{d\eta}{dx}\right)^2}{2}\right)^2 (6)$$

where $\xi(x,t)$ is the longitudinal displacement and the $\eta(x,t)$ is the transverse displacement of the string; thus the point in the undisturbed string has coordinates $[x,0]$ is moved to the point with coordinates $[x+\xi(x,t)+\eta(x,t)]$. The parameter $\tau_\circ$ represents the tension in the undisturbed string, λ is the product of Young's modulus and the cross-sectional area, and $\rho_\circ$ is the mass per unit length.

4-Classical Treatment of the Piano String Lagrangian Density

- **Euler-Lagrange Equation**

  Consider the Lagrangian density $\mathcal{L} = \mathcal{L}(\phi_r, \partial_\alpha \phi_r), r = 1,2,3,\ldots n$

  Using variation principle leads to Euler-Lagrange equation:

  $$\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_r)}\right) = 0 (7)$$

Applying this equation in two fields in the piano string$\left(\xi_\rho, \eta_\rho\right)$ Lagrangian density as

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}}\right) + \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\xi'}\right) - \frac{\partial\mathcal{L}}{\partial\xi} = 0 (8)$$

$$\ddot{\xi} - \frac{\lambda}{\rho_\circ}\left(\xi'' + \eta'\eta''\right) = 0 (9)$$

And

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\eta}}\right) + \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\eta'}\right) - \frac{\partial\mathcal{L}}{\partial\eta} = 0 (10)$$

$$\ddot{\eta} - \frac{\lambda}{\rho_\circ}\left[\left(\frac{2\tau_\circ}{\lambda} + \xi' + \frac{3}{2}\eta'^2\right)\eta'' + \eta'\xi''\right] = 0 (11)$$

- **Classical Hamiltonian Density**

  The Hamiltonian of the continuous system is given by

  $$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial(\dot{\phi}_r)}\dot{\phi}_r - \mathcal{L} (12)$$

Applying this definition to evaluate the Hamiltonian density resulting from Piano string Lagrangian density, we obtained

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\xi}}\dot{\xi} + \frac{\partial \mathcal{L}}{\partial \dot{\eta}}\dot{\eta} - \mathcal{L}$$

$$\mathcal{H} = \frac{\rho_\circ}{2}\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \frac{\lambda}{2}\left(\frac{\tau_\circ}{\lambda} + \xi' + \frac{\eta'^2}{2}\right)^2 (13)$$

- **Canonical Stress Tensor**

For invariant Lagrangian density, $\mathcal{L}$, the invariance under translation of time leads to conservation of energy. The invariance under translation of space leads to conservation of linear momentum; the invariance under rotational motion leads to conservation of angular momentum and the invariance under moving coordinates leads to linear motion of the center of mass. Emmy Noether constructed an expression that described these four cases as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial^{\mu} \phi}\partial^{\nu}\phi - g^{\mu\nu}\mathcal{L} \quad (14)$$

Applying this relation to construct $T^{\mu\nu}$, where $T^{\mu\nu}$ is called canonical stress tensor

$$T^{\mu\nu} = \begin{bmatrix} \rho\dot{\xi} g^{0\nu}\partial^{\nu}\xi - \lambda\left(\frac{\tau_\circ}{\lambda} + \xi' + \frac{\eta'^2}{2}\right) g^{1\nu}\partial^{\nu}\xi \\ + \xi\dot{\eta} g^{0\nu}\partial^{\nu}\eta - 2\lambda\left(\frac{\tau_\circ}{\lambda} + \xi' + \frac{\eta'^2}{2}\right)\eta' g^{1\nu}\partial^{\nu}\eta \end{bmatrix} (15)$$

as an example

$$T^{00} = \frac{\rho_\circ}{2}\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \frac{\lambda}{2}\left(\frac{\tau_\circ}{\lambda} + \xi' + \frac{\eta'^2}{2}\right)^2 = \mathcal{H} (16)$$

$$T^{11} = -\lambda\left(\frac{\tau_\circ}{\lambda} + \xi' + \frac{\eta'^2}{2}\right)(\xi' + 2\eta') (17)$$

$$T^{10} = \rho_\circ\left(\dot{\xi}^2 + \dot{\eta}^2\right) (18)$$

5-Fractional Formulation of Piano String Lagrangian Density

The Lagrangian density of the Piano string has the form as in equation ().Using the definition of the Riemann-Liouville fractional Derivative, this Lagrangian density takes the form

$$\mathcal{L} = \frac{\rho_\circ}{2}\left[\left({}_{+}\partial_t^{\alpha}\xi\right)^2 + \left({}_{+}\partial_t^{\alpha}\eta\right)^2\right] - \frac{\lambda}{2}\left[\frac{\tau_\circ}{2} + {}_{+}\partial_x^{\alpha}\xi + \frac{1}{2}\left({}_{+}\partial_x^{\alpha}\eta\right)^2\right]^2 (19)$$

Now let us start with a definition of fractional piano string Lagrangian density and use the generalization formula of the variational principle to obtain the equations of motion from piano string Lagrangian density.

Take the first field variable ξ, then

$$\frac{\partial \mathcal{L}}{\partial \xi} + \left[{}_{+}\partial_t^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_t^\alpha \xi)} \right) + {}_{+}\partial_x^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_x^\alpha \xi)} \right) \right] = 0 \quad (20)$$

Calculating these derivatives yields:

$$\frac{\partial \mathcal{L}}{\partial \xi} = 0 \quad (21)$$

$${}_{+}\partial_t^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_t^\alpha \xi)} \right) = \rho_\circ \, {}_{+}\partial_t^\alpha \, {}_{+}\partial_t^\alpha \xi \quad (22)$$

$${}_{+}\partial_x^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_x^\alpha \xi)} \right) = -\lambda \left[{}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \xi) + ({}_{+}\partial_x^\alpha \eta) \, {}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \eta) \right] \quad (23)$$

Substituting equations (21, 22, 23) in equation (20) we get

$$\rho_\circ \, {}_{+}\partial_t^\alpha \, {}_{+}\partial_t^\alpha \xi - \lambda \left[{}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \xi) + ({}_{+}\partial_x^\alpha \eta) \, {}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \eta) \right] = 0 \quad (24)$$

This represents the first equation of motion in fractional form for ξ field.

Now use the general formula to obtain other equations of motion for the other field η , then

$$\frac{\partial \mathcal{L}}{\partial \eta} + \left[{}_{+}\partial_t^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_t^\alpha \eta)} \right) + {}_{+}\partial_x^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_x^\alpha \eta)} \right) \right] = 0 \quad (25)$$

calculating these derivatives

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0 \quad (26)$$

$${}_{+}\partial_t^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_t^\alpha \eta)} \right) = \rho_\circ \, {}_{+}\partial_t^\alpha \, {}_{+}\partial_t^\alpha \eta \quad (27)$$

$${}_{+}\partial_x^\alpha \left(\frac{\partial \mathcal{L}}{\partial({}_{+}\partial_x^\alpha \eta)} \right) = -\lambda \left\{ \begin{matrix} \left[\frac{\tau_\circ}{2} + {}_{+}\partial_x^\alpha \xi + \frac{1}{2} ({}_{+}\partial_x^\alpha \eta)^2 \right] ({}_{+}\partial_x^\alpha \, {}_{+}\partial_x^\alpha \eta) \\ + \left[{}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \xi) + ({}_{+}\partial_x^\alpha \eta) \, {}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \eta) \right] {}_{+}\partial_x^\alpha \eta \end{matrix} \right\} \quad (28)$$

Substituting these equations in the main equation, we get:

$$\rho_\circ \, {}_{+}\partial_t^\alpha \, {}_{+}\partial_t^\alpha \eta - \lambda \left\{ \begin{matrix} \left[\frac{\tau_\circ}{2} + {}_{+}\partial_x^\alpha \xi + \frac{1}{2} ({}_{+}\partial_x^\alpha \eta)^2 \right] ({}_{+}\partial_x^\alpha \, {}_{+}\partial_x^\alpha \eta) \\ + \left[{}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \xi) + ({}_{+}\partial_x^\alpha \eta) \, {}_{+}\partial_x^\alpha ({}_{+}\partial_x^\alpha \eta) \right] {}_{+}\partial_x^\alpha \eta \end{matrix} \right\} = 0 \quad (29)$$

This equation represents the second equation of motion for second field η.

6-HAMILTONIAN FORMULATION OF FIELD WITHIN RIEMANN-LIOUVILLE FRACTIONAL DEFINTION

It is possible to derive the Hamiltonian equation in the fractional form for the field system such as integer order field. A continuous system with Lagrangian density given in terms of the dynamical field variables, generalized coordinates and its derivatives defined as [10-14]

$$\mathcal{L} = \mathcal{L}\left(\xi_\rho,\ {}_aD_t^\gamma \xi_\rho,\ {}_aD_x^\gamma \xi_\rho, \eta_\rho,\ {}_aD_t^\gamma \eta_\rho,\ {}_aD_x^\gamma \eta_\rho\right) \quad (30)$$

Now we construct the fractional Hamiltonian equations within Riemann-Liouville fractional derivative from a given Lagrangian density, define the fractional canonical momentum densities as

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial\left({}_+\partial_t^\alpha \phi_i\right)} \quad (31)$$

$$\pi_{\alpha_\xi} = \frac{\partial \mathcal{L}}{\partial\left({}_+\partial_t^\alpha \xi\right)} = \rho_\circ \dot{\xi} \quad (32)$$

$$\pi_{\alpha_\eta} = \frac{\partial \mathcal{L}}{\partial\left({}_+\partial_t^\alpha \eta\right)} = \rho_\circ \dot{\eta} \quad (33)$$

Then the Hamiltonian[13]

$$\mathcal{H} = \sum_{k=0} (\pi_\alpha)\ {}_+\partial_k^\alpha \phi - \mathcal{L} \quad (34)$$

$$\mathcal{H} = \left(\pi_{\alpha_\xi}\right)\ {}_+\partial_t^\alpha \xi + \left(\pi_{\alpha_\eta}\right)\ {}_+\partial_t^\alpha \eta - \mathcal{L} \quad (35)$$

Take the total derivative of both sides

$$dH = d\pi_{\alpha_\xi}\ {}_+\partial_t^\alpha \xi + d\pi_{\alpha_\eta}\ {}_+\partial_t^\alpha \eta - \frac{\partial \mathcal{L}}{\partial \xi_\rho} d\xi_\rho - \frac{\partial \mathcal{L}}{\partial \eta_\rho} d\eta_\rho - \frac{\partial \mathcal{L}}{\partial\left({}_a\partial_x^\alpha \xi_\rho\right)} d\left({}_a\partial_x^\alpha \xi_\rho\right) - \frac{\partial \mathcal{L}}{\partial\left({}_a\partial_x^\alpha \eta_\rho\right)} d\left({}_a\partial_x^\alpha \eta_\rho\right) - \frac{\partial \mathcal{L}}{\partial t} dt \quad (36)$$

But the Hamiltonian is function of the form:

$$H = H\left(\xi_\rho, \eta_\rho, \pi_{\alpha_\xi}, \pi_{\alpha_\eta}, t,\ {}_a\partial_x^\alpha \xi_\rho,\ {}_a\partial_x^\alpha \eta_\rho\right) \quad (37)$$

So, the total differential of the Hamiltonian takes the form

$$dH=\frac{\partial H}{\partial \xi_\rho}d\xi_\rho+\frac{\partial H}{\partial \eta_\rho}d\eta_\rho+\frac{\partial H}{\partial \pi_{\alpha_\xi}}d\pi_{\alpha_\xi}+\frac{\partial H}{\partial \pi_{\alpha_\eta}}d\pi_{\alpha_\eta}+\frac{\partial H}{\partial t}dt+\frac{\partial H}{\partial\ {}_a\partial_x^\alpha\xi_\rho}d\ {}_a\partial_x^\alpha\xi_\rho+\frac{\partial H}{\partial\ {}_a\partial_x^\alpha\eta_\rho}d\ {}_a\partial_x^\alpha\eta_\rho \quad (38)$$

Compare the last two equations to get the Hamiltonian equations of motion

$$\left[\frac{\partial H}{\partial t}=-\frac{\partial \mathcal{L}}{\partial t}\left[\begin{matrix}\frac{\partial H}{\partial \xi_\rho}=-\frac{\partial \mathcal{L}}{\partial \xi_\rho}\\ \frac{\partial H}{\partial \eta_\rho}=-\frac{\partial \mathcal{L}}{\partial \eta_\rho}\end{matrix}\right.\left[\begin{matrix}\frac{\partial H}{\partial \pi_{\alpha_\xi}}={}_+\partial_t^\alpha\xi\\ \frac{\partial H}{\partial \pi_{\alpha_\eta}}={}_+\partial_t^\alpha\eta\end{matrix}\right.\left[\begin{matrix}\frac{\partial H}{\partial\ {}_a\partial_x^\alpha\xi_\rho}=-\frac{\partial \mathcal{L}}{\partial(\ {}_a\partial_x^\alpha\xi_\rho)}\\ \frac{\partial H}{\partial\ {}_a\partial_x^\alpha\eta_\rho}=-\frac{\partial \mathcal{L}}{\partial(\ {}_a\partial_x^\alpha\eta_\rho)}\end{matrix}\right.\right. \quad (39)$$

$$\mathcal{H}=\frac{\rho_\circ}{2}\left[\left({}_+\partial_t^\alpha\xi\right)^2+\left({}_+\partial_t^\alpha\eta\right)^2\right]+\frac{\lambda}{2}\left[\frac{\tau_\circ}{2}+{}_+\partial_x^\alpha\xi+\frac{1}{2}\left({}_+\partial_x^\alpha\eta\right)^2\right]^2 \quad (40)$$

7-Noether's theorem in fractional form

Noether's theorem is one of the most important theorems in mechanics which tells us that all conservation laws related to the invariance of the action under a family of transformations. Assume the system is invariant under translation in space-time. For fractional Piano string, Lagrangian density is function of

$$\mathcal{L}=\mathcal{L}\left(\xi_\rho,\ {}_aD_t^\gamma\xi_\rho,\ {}_aD_x^\gamma\xi_\rho,\eta_\rho,\ {}_aD_t^\gamma\eta_\rho,\ {}_aD_x^\gamma\eta_\rho\right) \quad (41)$$

The variational of $\mathcal{L}$ can be written as[13-14]

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\xi_\rho}\delta\xi_\rho+\frac{\partial\mathcal{L}}{\partial\eta_\rho}\delta\eta_\rho+\frac{\partial\mathcal{L}}{\partial\ {}_+\partial_t^\alpha\xi}\delta\ {}_+\partial_t^\alpha\xi+\frac{\partial\mathcal{L}}{\partial\ {}_+\partial_t^\alpha\eta}\delta\ {}_+\partial_t^\alpha\eta+\frac{\partial\mathcal{L}}{\partial\ {}_+\partial_x^\alpha\xi}\delta\ {}_+\partial_x^\alpha\xi+\frac{\partial\mathcal{L}}{\partial\ {}_+\partial_x^\alpha\eta}\delta\ {}_+\partial_x^\alpha\eta \quad (42)$$

Using Euler-Lagrange equation defined in equations (20), we obtain:

$$\partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\ {}_a\partial_t^\alpha\xi_\rho)}\ {}_a\partial_t^\beta\xi_\rho+\frac{\partial\mathcal{L}}{\partial(\ {}_a\partial_t^\alpha\eta_\rho)}\ {}_a\partial_t^\beta\eta_\rho-g^{\alpha\beta}\mathcal{L}\right]=0 \quad (43)$$

Which can be written like $[\partial_{\alpha} T^{\mu\nu} = 0]$, where $T^{\mu\nu}$ fractional stress tensor for two ξ_{ρ}, η_{ρ} fields, and $g^{\mu\nu}$ metric tensor $(g^{\mu\nu} = 1, -1, -1, -1)$

Applying equation (43) enables us to determine the scalar fractional canonical stress tensor, as follows

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left({}_{a}\partial_{x_{\mu}}^{\alpha} \xi_{\rho}\right)}\ {}_{a}\partial_{x_{\nu}}^{\beta} \xi_{\rho} + \frac{\partial \mathcal{L}}{\partial\left({}_{a}\partial_{x_{\mu}}^{\alpha} \eta_{\rho}\right)}\ {}_{a}\partial_{x_{\nu}}^{\beta} \eta_{\rho} - g^{\mu\nu}\mathcal{L} \quad (44)$$

$$T^{\mu\nu} = \begin{bmatrix} \rho\left({}_{+}\partial_{t}^{\alpha} \xi\right) g^{0\nu}\left({}_{+}\partial_{x}^{\alpha} \xi\right) - \lambda\left(\frac{\tau_{\circ}}{\lambda} + \left({}_{+}\partial_{x}^{\alpha} \xi\right) + \frac{\left({}_{+}\partial_{x}^{\alpha} \eta\right)^{2}}{2}\right) g^{1\nu}\left({}_{+}\partial_{x}^{\alpha} \xi\right) \\ + \xi\left({}_{+}\partial_{t}^{\alpha} \eta\right) g^{0\nu}\left({}_{+}\partial_{x}^{\alpha} \eta\right) - 2\lambda\left(\frac{\tau_{\circ}}{\lambda} + {}_{+}\partial_{x}^{\alpha} \xi + \frac{\left({}_{+}\partial_{x}^{\alpha} \eta\right)^{2}}{2}\right) \eta' g^{1\nu}\left({}_{+}\partial_{x}^{\alpha} \eta\right) \end{bmatrix} \quad (45)$$

As an example

$$T^{00} = \frac{\rho_{\circ}}{2}\left[\left({}_{+}\partial_{t}^{\alpha} \xi\right)^{2} + \left({}_{+}\partial_{t}^{\alpha} \eta\right)^{2}\right] + \frac{\lambda}{2}\left[\frac{\tau_{\circ}}{2} + {}_{+}\partial_{x}^{\alpha} \xi + \frac{1}{2}\left({}_{+}\partial_{x}^{\alpha} \eta\right)^{2}\right]^{2} \quad (46)$$

Which is of similar value to the fractional Hamiltonian result from Piano string Lagrangian density.

$$T^{10} = \rho_{\circ}\left[\left({}_{a}\partial_{t}^{\alpha} \xi_{\rho}\right)^{2} + \left({}_{a}\partial_{t}^{\alpha} \eta_{\rho}\right)^{2}\right] \quad (47)$$

$$T^{11} = -\lambda\left(\frac{\tau_{\circ}}{\lambda} + \left({}_{a}\partial_{x}^{\alpha} \xi_{\rho}\right) + \frac{\left({}_{a}\partial_{x}^{\alpha} \eta_{\rho}\right)^{2}}{2}\right)\left({}_{a}\partial_{x}^{\alpha} \xi_{\rho} + 2\ {}_{a}\partial_{x}^{\alpha} \eta_{\rho}\right) \quad (48)$$

The connection of time-time and space-time components of $T^{\mu\nu}$ with the field energy and momentum density suggests that there is a covariant generalization of the conservation laws. The covariant form of the conservation law ${}_{a}\partial_{x_{\mu}}^{\alpha} T^{\mu\nu} = 0$ leads to conservation of total energy and momentum upon integration over the spaces.

8-CONCLUSION

For a given Lagrangian density, we obtained that the fractional Euler-Lagrange equation and fractional Hamiltonian equation of motion lead to the same result.The classical Euler-Lagrange equation was obtained as a particular case of the fractional formulation. The fractional conserved quantities are derived from the Piano string Lagrangian density since it is invariant under space time transformation.

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